

SUBNORMAL CLOSURE OF A HOMOMORPHISM

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ABSTRACT. Let $\varphi: \Gamma \rightarrow G$ be a homomorphism of groups. In this paper we introduce the notion of a subnormal map (the inclusion of a subnormal subgroup into a group being a basic prototype). We then consider factorizations $\Gamma \xrightarrow{\psi} M \xrightarrow{n} G$ of φ , with n a subnormal map. We search for a universal such factorization. When Γ and G are finite we show that such universal factorization exists: $\Gamma \rightarrow \Gamma_\infty \rightarrow G$, where Γ_∞ is a hypercentral extension of the subnormal closure \mathcal{C} of $\varphi(\Gamma)$ in G (i.e. the kernel of the extension $\Gamma_\infty \rightarrow \mathcal{C}$ is contained in the hypercenter of Γ_∞). This is closely related to the a relative version of the Bousfield-Kan \mathbb{Z} -completion tower of a space. The group Γ_∞ is the inverse limit of the normal closures tower of φ introduced by us in a recent paper. We prove several stability and finiteness properties of the tower and its inverse limit Γ_∞ .

1. INTRODUCTION

Throughout this note $\varphi: \Gamma \rightarrow G$ is a homomorphism of groups. In a previous paper [FS1] we considered the notion of the *free normal closure* Γ^φ of φ , (related to [BH, BHS]) and the *free normal closures tower* $\{\Gamma_{i,\varphi}\}_{i=1}^\infty$ of φ (see equation (1.1) below). In this paper we study the behavior and the properties of the inverse limit $\Gamma_{\infty,\varphi}$ of the tower of free normal closures of φ . This tower generalizes and connects the quotients of the lower central series $\Gamma/\gamma_i(\Gamma)$, $i = 1, 2, 3, \dots$, gotten here for $G = 1$, and the descending series of successive normal closures of a subgroup H of G (see §2). Notice that in the case $G = 1$, the group $\Gamma_{\infty,\varphi}$ is the nilpotent completion of Γ . Thus for an arbitrary map φ , the group $\Gamma_{\infty,\varphi}$ can be thought of as a relative nilpotent completion associated with a homomorphism rather than with a group. Some of the results here support this point of view.

Let us recall from [FS1, section 4] that the normal closures tower (we often omit the word “free”) associated with a homomorphism φ is a tower of groups as follows:

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$$(1.1) \quad \begin{array}{c} \Gamma \\ \searrow \varphi_1 \quad \searrow \varphi_2 \quad \searrow \varphi_k \quad \searrow \varphi_{k+1} \\ \dots \longrightarrow \Gamma_{k+1} \xrightarrow{\overline{\varphi_k}} \Gamma_k \longrightarrow \dots \longrightarrow \Gamma_2 \xrightarrow{\overline{\varphi_1}} \Gamma_1 = G \end{array}$$

where $\overline{\varphi_i}$ is a normal map (our terminology for a crossed module), and where $\Gamma_{i+1} = \Gamma_{i+1, \varphi}$ is the (free) normal closure $\Gamma_i^{\varphi_i}$ of the map $\Gamma \xrightarrow{\varphi_i} \Gamma_i$, for all $i \geq 1$. The group $\Gamma_\infty = \Gamma_{\infty, \varphi}$, the stability of the tower $\{\Gamma_{i, \varphi}\}$ and its relation to the groups Γ, G and the map φ , are thus the main topics of the present study.

One way to think about the above tower is that it represents an attempt to factor the map φ , in a universal way, into a composition of “simpler maps” (i.e. normal maps). This, of course, cannot be done in general, and $\Gamma_{\infty, \varphi}$ is a kind of “hybrid” of Γ and G giving a factorization $\Gamma \rightarrow \Gamma_{\infty, \varphi} \rightarrow G$. Passing to topological spaces via the classifying space construction we get a map $B\Gamma \rightarrow BG$. The present result shows that for finite groups Γ, G this map has a finite “relative” Bousfield-Kan tower of principal fibrations [BK] whose fibres are, in general, neither connected nor nilpotent groups:

$$B\Gamma_\infty = B\Gamma_n \rightarrow B\Gamma_{n-1} \rightarrow \dots \rightarrow B\Gamma_2 \rightarrow BG;$$

with a terminal term $B\Gamma_\infty = B\Gamma_n \rightarrow BG$ being *the universal space*, under $B\Gamma$ for which there is such a tower of principal fibrations. This raises the question of finding such universal decompositions of more general maps of spaces $X \rightarrow Y$. Notice that we have an induced map $\varphi_\infty: \Gamma \rightarrow \Gamma_\infty$; we often ask how far is this map from an isomorphism.

One quick corollary of our main results concerns a map of nilpotent groups:

Theorem 1. *Let $\varphi: \Gamma \rightarrow G$ be a homomorphism of nilpotent groups, then $\varphi_\infty: \Gamma \rightarrow \Gamma_\infty$ is an isomorphism.*

Theorem 1 is Corollary 2.13(2). This corollary can be viewed as a version for any map of nilpotent groups, of the well known property that any subgroup of a nilpotent group is subnormal, i.e. equal to its subnormal closure.

In this spirit we have for any homomorphism of finite groups:

Theorem 2. *Let $\varphi: \Gamma \rightarrow G$ be a homomorphism of finite groups. Then φ_∞ induces an isomorphism of the descending central series quotients, $\Gamma/\gamma_i(\Gamma) \cong \Gamma_\infty/\gamma_i(\Gamma_\infty)$, for all $i \geq 1$.*

Theorem 2 is proved in Proposition 2.14. We note that by previous results, this is certainly not true for a general map of groups, e.g., when Γ is a free group with infinitely many generators and $G = 1$.

Our next result extends a partial result in [FS1] to the general finite case. An estimate of the size of Γ_∞ is given below in Theorem 2.11.

Theorem 3. *If $\varphi: \Gamma \rightarrow G$ be a homomorphism of finite groups then $\Gamma_{\infty, \varphi}$ is a finite group.*

We note that even when φ is the trivial map between two finite cyclic groups, the groups $\Gamma_{i, \varphi}$ grow indefinitely in size with i , but their inverse limit is finite, and is isomorphic to the domain in this case (Theorem 1).

Recall that for a finite group G and a subgroup $H \leq G$, the *subnormal closure* of H in G is the smallest subnormal subgroup of G containing H . Thus Theorem 3 is again an extension to any map of finite groups, of the trivial observation that the subnormal closure of a subgroup of a finite group is well defined and finite. This result complements the dual result proved in [FS1] for the tower of injective normalizers of a map of finite groups.

The next theorem characterizes, for maps of finite groups, the factorization $\Gamma \rightarrow \Gamma_{\infty} \rightarrow G$ as a universal one among all subnormal factorizations (see below) the proof is given in §3:

The following is a basic definition:

Definition 4. A *subnormal map* $n: M \rightarrow G$, is a homomorphism such that there exists a finite series of normal maps $n_i: M_{i+1} \rightarrow M_i$, $1 \leq i \leq k$, with $M_1 = G$, whose composition is n .

$$M_{k+1} \xrightarrow{n_k} M_k \longrightarrow \dots \longrightarrow M_2 \xrightarrow{n_1} G.$$

$\underbrace{\hspace{10em}}_n$

Notice that the image subgroup $n(M_{k+1})$ is subnormal in G .

For example, any map of nilpotent groups is a subnormal map.

Theorem 5. *If $\varphi: \Gamma \rightarrow G$ be a homomorphism of finite groups then the factorization $\Gamma \rightarrow \Gamma_{\infty, \varphi} \rightarrow G$ is the universal initial factorization of φ , among all factorizations $\Gamma \rightarrow S \rightarrow G$ of φ , with the right map $S \rightarrow G$ being a subnormal map; namely, it maps uniquely to every such factorization via $\Gamma_{\infty, \varphi} \rightarrow S$.*

Main tools. Here we sum-up the main technical tools for proving the above results. They consist of showing that the limiting group $\Gamma_{\infty, \varphi}$ does not change up to a canonical isomorphism, when one changes the domain or range of φ in certain controlled ways.

First we consider changing the range via factoring our map φ through any subnormal map $M \rightarrow G$.

The following theorem is one of our main tools showing we can perform the above mentioned replacement:

Theorem 6. *Let $\varphi: \Gamma \rightarrow G$ be a homomorphism and let $\Gamma \xrightarrow{\psi} M \xrightarrow{n} G$ be a factorization of φ such that $n: M \rightarrow G$ is a subnormal map. Then $\Gamma_{\infty, \varphi} \cong \Gamma_{\infty, \psi}$.*

Theorem 6 is proved right after Proposition 2.4.

The next result shows that one can factor out a certain portion \mathcal{K} of the kernel K of φ and still obtain that $\Gamma_{\infty, \varphi} \cong (\Gamma/\mathcal{K})_{\infty, \rho}$, where $\rho: \Gamma/\mathcal{K} \rightarrow G$ is the map induced by φ .

To define \mathcal{K} , define the *descending series of successive commutators of K with Γ* by $K_1 = K$, $K_2 = [\Gamma, K]$, and in general $K_{i+1} = [\Gamma, K_i]$. If this series terminates after a finite number of steps, we let \mathcal{K} be the final member of this series.

Proposition 7. *Let $\varphi: \Gamma \rightarrow G$ be a homomorphism, and suppose that the descending series of successive commutators of $K = \ker \varphi$ with Γ terminates after a finite number of steps. Let \mathcal{K} be the terminal member of this series. Then $\Gamma_{\infty, \varphi} \cong (\Gamma/\mathcal{K})_{\infty, \rho}$, where $\rho: \Gamma/\mathcal{K} \rightarrow G$ is the map induced by φ .*

Proposition 7 is Proposition 2.9(2). Notice that in the notation of Proposition 7, the kernel of ρ is K/\mathcal{K} and hence $\ker \rho$ is contained in some member of the ascending central series of Γ/\mathcal{K} . In Example 4.1 we show however that the nilpotency class of K/\mathcal{K} cannot be bounded.

2. EQUIVALENCE OF NORMAL CLOSURES TOWERS

In this section we prove the results of the introduction, and analyze further the group $\Gamma_{\infty, \varphi}$. Our main tool is to compare the normal closures tower $\{\Gamma_{i, \varphi}\}$ to towers $\{\Lambda_{i, \rho}\}$, for various homomorphism $\rho: \Lambda \rightarrow H$, which are, of course, related to φ .

As noted above, throughout this paper $\varphi: \Gamma \rightarrow G$ is a fixed homomorphism of groups. We use the following notation. $K = \ker \varphi$ and Γ^φ is the (free) normal closure of φ . This, we recall, is the universal factorization $\Gamma \xrightarrow{c_\varphi} \Gamma^\varphi \xrightarrow{\bar{\varphi}} G$ of φ with the right map being normal.

As in diagram (1.1), $\{\Gamma_i\} = \{\Gamma_{i, \varphi}\}$ is the normal closures tower of φ , and $\varphi_i, \bar{\varphi}_i$ are as in diagram (1.1). $\Gamma_\infty = \varprojlim \Gamma_i$ and $\varphi_\infty: \Gamma \rightarrow \Gamma_\infty$ is the natural map.

Terminology 2.1. *Let $k \geq 0$ be an integer, and let $\{H_i\}_{i=k}^\infty$ be a decreasing or increasing series of groups. We say that $\{H_i\}$ **terminates** if there exists an integer $t \geq k$, such that $H_t = H_{t+1} = H_{t+2} = \dots$. In this case we call H_t the **terminal member** of the series and say that the series terminates at H_t .*

Recall from [R, p. 385] the notion of the *series of successive normal closures—in the usual sense—of a subgroup $H \leq G$ in G* . This is the decreasing series defined by $C_0 = G$, $C_1 = \langle H^{C_0} \rangle$, and in general $C_{i+1} = \langle H^{C_i} \rangle$.

Notation 2.2. (1) The series of successive normal closures of $\varphi(\Gamma)$ in G will be denoted $G = C_0, C_1, C_2, \dots$. If this series terminates, then we denote by \mathcal{C} its terminal member.

(2) Let $K := \ker \varphi$ and define the decreasing *series of successive commutators of K with Γ* by $K_1 = K$, $K_2 = [\Gamma, K]$, and in general $K_{i+1} = [\Gamma, K_i]$. If this series terminates, then we denote by \mathcal{K} its terminal member.

Next $H = \gamma_1(H), \gamma_2(H), \dots$ denotes the descending central series of the group H . If this series terminates, then $\gamma_\infty(H)$ denotes the terminal member of this series. Finally, the

ascending central series of H is denoted $1 = Z_0(H), Z_1(H), \dots$, and the same convention as above applies for the notation $Z_\infty(H)$.

We refer the reader to [FS1, section 2] for the notions of a *normal map* and of *normal morphism*, where references to previous work on this subject in given. In [FS1, section 3] the reader will find some basic properties of Γ^φ and in [FS1, section 4] some basic properties of the normal closures tower of φ .

Let us start by recalling the naturality of the normal closures tower.

Lemma 2.3. *Any commutative diagram*

$$(2.1) \quad \begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & G \\ \mu \downarrow & & \downarrow \eta \\ \Gamma' & \xrightarrow{\varphi'} & G' \end{array}$$

*induces a commutative diagram between the towers of normal closures of φ and φ' :
Namely if we set $\Gamma_1 := G$, $\Gamma'_1 = G'$ and $\eta_1 := \eta$, we have:*

$$(2.2) \quad \begin{array}{ccccccc} \Gamma & \longrightarrow & \Gamma_{i+1} & \xrightarrow{\overline{\varphi_i}} & \Gamma_i & \longrightarrow & \dots \longrightarrow \Gamma_1 \\ \mu \downarrow & & \eta_{i+1} \downarrow & & \downarrow \eta_i & & \downarrow \eta_1 \\ \Gamma' & \longrightarrow & \Gamma'_{i+1} & \xrightarrow{\overline{\varphi'_i}} & \Gamma'_i & \longrightarrow & \dots \longrightarrow \Gamma'_1 \end{array}$$

Thus there is a canonical map $\eta_\infty: \Gamma_\infty \rightarrow \Gamma'_\infty$ and equalities: $\varphi_i \circ \eta_i = \mu \circ \varphi'_i$, for all $i \geq 1$.

Proof. Take in diagram (2.2) of [FS1], $M' = \Gamma'_2$, $\psi' = \varphi'_2$ and $n' = \overline{\varphi'_1}$, and use the universality properties of Γ_2 to obtain η_2 . Then proceed in this manner replacing each time Γ_i , Γ'_i , with Γ_{i+1} , Γ'_{i+1} respectively. \square

The following Proposition will be used in the proof of Theorem 6 of the introduction. It addresses the replacement of G by the domain of any subnormal map $M \rightarrow G$ that factors our map φ . In that case the map of towers (2.3) is in fact a pro-isomorphism of towers:

Proposition 2.4. *Let $\Gamma \xrightarrow{\psi} M \xrightarrow{n} G$ be a factorization of φ with n a normal map. Then the above 2.2 commutative diagram extends to a natural commutative diagram in which the upper (resp. lower) row is the normal closures tower of $\Gamma \xrightarrow{\psi} M$ (resp. $\Gamma \xrightarrow{\varphi} G$):*

$$(2.3) \quad \begin{array}{ccccccccccc} \Gamma & \longrightarrow & \Delta_i & \xrightarrow{\overline{\psi}_{i-1}} & \Delta_{i-1} & \longrightarrow & \dots & \longrightarrow & \Delta_3 & \xrightarrow{\overline{\psi}_2} & \Delta_2 & \xrightarrow{\overline{\psi}_1} & \Delta_1 = M \\ \downarrow = & \nearrow \rho_i & \downarrow \mu_i & \nearrow \rho_{i-1} & \downarrow \mu_{i-1} & & & & \downarrow \mu_3 & \nearrow \rho_2 & \downarrow \mu_2 & \nearrow \rho_1 & \downarrow \mu_1 = n \\ \Gamma & \longrightarrow & \Gamma_{i+1} & \xrightarrow{\overline{\varphi}_i} & \Gamma_i & \xrightarrow{\overline{\varphi}_{i-1}} & \Gamma_{i-1} & \longrightarrow & \dots & \longrightarrow & \Gamma_3 & \xrightarrow{\overline{\varphi}_2} & \Gamma_2 & \xrightarrow{\overline{\varphi}_1} & \Gamma_1 = G \end{array}$$

Proof. We need the following lemma which is an obvious analog of the similar situation for two normal subgroup of the same group:

Lemma 2.5. *Consider the commutative diagram*

$$\begin{array}{ccc} M_1 & \xrightarrow{\mu} & M_2 \\ & \searrow n_1 & \swarrow n_2 \\ & G & \end{array}$$

where n_i are normal maps, for $i = 1, 2$, and μ is a normal morphism.

Proof. Define an action of M_2 on M_1 by $m_1^{m_2} = m_1^{n_2(m_2)}$, $m_i \in M_i$. Then it is immediate that μ becomes a normal map with the above normal structure. \square

As usual let $\Gamma_1 := G$ and $\varphi_1 := \varphi$. Consider the normal closures tower corresponding to ψ . So the groups Γ_i in diagram (1.1) are replaced by Δ_i and the maps $\varphi_i, \bar{\varphi}_i$ are replaced by $\psi_i, \bar{\psi}_i$, respectively. Thus $\Delta_1 = M$ and $\psi_1 = \psi$. Set $\mu_1 = n$, thus $\mu_1: \Delta_1 \rightarrow \Gamma_1$ is a normal map.

We show that there are normal maps μ_i, ρ_i , $i \geq 1$, as in diagram (2.3) such that

$$(2.4) \quad \varphi_{i+1} \circ \rho_i = \psi_i \text{ and } \psi_i \circ \mu_i = \varphi_i, \text{ for all } i \geq 1.$$

Now since $\Gamma \xrightarrow{\psi_1} \Delta_1 \xrightarrow{\mu_1} \Gamma_1$ is a factorization of φ_1 , by the universality property of Γ_2 , there exists a normal morphism $\rho_1: \Gamma_2 \rightarrow \Delta_1$ for the lower right triangle, and such that $\varphi_2 \circ \rho_1 = \psi_1$. By Lemma 2.5, ρ_1 is a normal map.

Let now $i \geq 2$, and suppose that ρ_{i-1} and μ_{i-1} were defined, they are normal maps, diagram (2.3) is commutative up to the $(i-1)$ -step, and equation (2.4) holds. By equation (2.4) we have $\varphi_i \circ \rho_{i-1} = \psi_{i-1}$. By the universality of Δ_i , and since ρ_{i-1} is a normal map, there exists a map $\mu_i: \Delta_i \rightarrow \Gamma_i$ such that

- (a) $\psi_i \circ \mu_i = \varphi_i$.
- (b) $\mu_i \circ \rho_{i-1} = \bar{\varphi}_i$, and μ_i is a normal morphism for the right triangle.

By Lemma 2.5, μ_i is a normal map.

Next, by the universality property of Γ_{i+1} , by (a), and since μ_i is a normal map, there exists a normal morphism $\rho_i: \Gamma_{i+1} \rightarrow \Delta_i$ for the lower right triangle, this triangle is commutative ($\rho_i \circ \mu_i = \bar{\varphi}_i$) and $\varphi_{i+1} \circ \rho_i = \psi_i$. Again by Lemma 2.5, ρ_i is a normal map. This completes the induction step and the proof of the proposition. \square

Proof of Theorem 6. Consider the subnormal series of normal maps

$$M_{k+1} \xrightarrow{n_k} M_k \longrightarrow \dots \longrightarrow M_2 \xrightarrow{n_1} G = M_1$$

$\underbrace{\hspace{10em}}_n$

where $k \geq 1$, $M_{k+1} = M$, $M_1 = G$, and n_t is a normal map, for all $1 \leq t \leq k$. Let $\psi_t: \Gamma \rightarrow M_t$ be the maps defined by $\psi_{k+1} = \psi$, and for $1 \leq t \leq k$, let $\psi_t = \psi \circ n_k \circ \dots \circ n_t$ (so $\psi_1 = \varphi$).

The universality property of the inverse limit and Proposition 2.4 imply that $\Gamma_{\infty, \psi_t} \cong \Gamma_{\infty, \psi_{t+1}}$, for all $1 \leq t \leq k$, so since $\Gamma_{\infty, \varphi} = \Gamma_{\infty, \psi_1}$ and $\Gamma_{\infty, \psi} = \Gamma_{\infty, \psi_{k+1}}$, the theorem holds. \square

As a corollary to Theorem 6 we get

Proposition 2.6. (1) Let H be a subnormal subgroup of G containing $\varphi(\Gamma)$. Then $\Gamma_{\infty, \varphi} \cong \Gamma_{\infty, \psi}$, where $\psi: \Gamma \rightarrow H$ is the restriction of φ in the range;
 (2) if the series $\{C_i\}$ terminates, then $\Gamma_{\infty, \varphi} = \Gamma_{\infty, \psi}$, where $\psi: \Gamma \rightarrow \mathcal{C}$ is the restriction of φ in the range.

Proof. Let $n: H \hookrightarrow G$ be the inclusion map. Then n is a subnormal map, and $\Gamma \xrightarrow{\psi} H \xrightarrow{n} G$ is a factorization of φ . Hence (1) follows from Theorem 6, and (2) is immediate from (1). \square

We now turn our attention to the kernel K .

Lemma 2.7. Let $L \leq \ker c_\varphi$ be a normal subgroup of Γ . Let $\rho: \Gamma/L \rightarrow G$ be the homomorphism induced by φ , and let $\beta: \Gamma/L \rightarrow \Gamma^\varphi$ be the homomorphism induced by c_φ . Then Γ^φ and $(\Gamma/L)^\rho$ are naturally isomorphic.

Proof. Consider the following commutative diagram. We show that u is an isomorphism whose inverse is v .

$$(2.5) \quad \begin{array}{ccccc} & & \Gamma^\varphi & & \\ & \nearrow c_\varphi & \downarrow u & \nwarrow \bar{\varphi} & \\ & & (\Gamma/L)^\rho & & \\ & \nearrow c_\rho & \downarrow v & \nwarrow \bar{\rho} & \\ & & \Gamma^\varphi & & \\ \Gamma & \xrightarrow{\alpha} \Gamma/L & \xrightarrow{\beta} \Gamma^\varphi & \xrightarrow{\bar{\rho}} G & \\ & \searrow \varphi & & & \end{array}$$

here $\alpha \circ \rho = \varphi$, $\alpha \circ \beta = c_\varphi$, and u and v are the unique normal morphisms obtained from the universality properties of Γ^φ and $(\Gamma/L)^\rho$ respectively.

Notice that $\alpha \circ \beta \circ \bar{\varphi} = c_\varphi \circ \bar{\varphi} = \varphi = \alpha \circ \rho$. Since α is surjective we see that $\beta \circ \bar{\varphi} = \rho$. By the universality property of $(\Gamma/L)^\rho$ we get the map v . Also, $(\alpha \circ c_\rho) \circ \bar{\rho} = \alpha \circ \rho = \varphi$. By the universality property of Γ^φ we get the map u .

Next we have $c_\varphi \circ u \circ v = \alpha \circ c_\rho \circ v = \alpha \circ \beta = c_\varphi$. Also $u \circ v \circ \bar{\varphi} = u \circ \bar{\rho} = \bar{\varphi}$. Hence $u \circ v = 1_{\Gamma^\varphi}$, by the uniqueness in the universality property of Γ^φ .

Further, $\alpha \circ c_\rho \circ v \circ u = \alpha \circ \beta \circ u = c_\varphi \circ u = \alpha \circ c_\rho$. Since α is surjective, $c_\rho \circ v \circ u = c_\rho$. Also $v \circ u \circ \bar{\rho} = v \circ \bar{\varphi} = \bar{\rho}$. So, as above, $v \circ u$ is the identity map of $(\Gamma/L)^\rho$. \square

As a corollary we get

Corollary 2.8. Let $L \leq \ker c_\varphi$ be a normal subgroup of Γ , and let $\rho: \Gamma/L \rightarrow G$ be the homomorphism induced by φ . Then $\Gamma/L \xrightarrow{\beta} \Gamma^\varphi \xrightarrow{\bar{\varphi}} G$, where β and $\bar{\varphi}$ are as in diagram (2.5), is the universal normal decomposition of ρ .

Proof. This follows from Lemma 2.7, because in diagram (2.5) we may take that $c_\rho = \beta$, $(\Gamma/L)^\rho = \Gamma^\varphi$ and $\bar{\rho} = \bar{\varphi}$. \square

In particular we get Proposition 7 of the introduction as part (2) of the following:

Proposition 2.9. *Let $L \leq \Gamma$ be a subgroup with $L \leq \ker \varphi_i$, for all integers $i \geq 1$. Let $\rho: \Gamma/L \rightarrow G$ be the map induced by φ , let $\{(\Gamma/L)_i\} = \{(\Gamma/L)_{i,\rho}\}$ be the normal closures tower of ρ , and let $\rho_i, \bar{\rho}_i$ be the maps for the tower $\{(\Gamma/L)_i\}$. Then*

- (1) *there is a natural isomorphism $\Gamma_i \cong (\Gamma/L)_i$, where $\rho_i: \Gamma/L \rightarrow (\Gamma/L)_i$ is the map induced by φ_i , and $\bar{\varphi}_i = \bar{\rho}_i$, for all $i \geq 1$. In particular $\Gamma_\infty \cong (\Gamma/L)_\infty$;*
- (2) *if the series $\{K_i\}$ terminates at \mathcal{K} , then (1) above holds for $L := \mathcal{K}$.*

Proof. Part (1) is immediate from Corollary 2.8. For (2) note that $\varphi_{i+1}(\ker \varphi_i) \leq \ker \bar{\varphi}_i \leq Z(\Gamma_{i+1})$. It follows by induction on i , that $K_i \leq \ker \varphi_i$, for all $i \geq 1$. This implies that if the series $\{K_i\}$ terminates, then $\mathcal{K} \leq \ker \varphi_i$, for all $i \geq 1$, so (2) follows from (1). \square

Combining Propositions 2.9(2) and 2.6(2) we get:

Corollary 2.10. *Assume that both the series $\{C_i\}$ and the series $\{K_i\}$ terminate at \mathcal{C} and \mathcal{K} respectively. Then there is an isomorphism $\Gamma_{\infty,\varphi} \cong (\Gamma/\mathcal{K})_{\infty,\psi}$, where $\psi: \Gamma/\mathcal{K} \rightarrow \mathcal{C}$, is the map induced by φ .*

Proof. By Proposition 2.9(2), $\Gamma_{\infty,\varphi} = (\Gamma/\mathcal{K})_{\infty,\rho}$, where $\rho: \Gamma/\mathcal{K} \rightarrow G$ is the map induced by φ . Then, by Proposition 2.6(2) (with Γ/\mathcal{K} in place of Γ), $(\Gamma/\mathcal{K})_{\infty,\rho} = (\Gamma/\mathcal{K})_{\infty,\psi}$. \square

The following theorem proves in particular the assertion of Theorem 3 of the introduction.

Theorem 2.11. *Let $\varphi: \Gamma \rightarrow G$ be a homomorphism of finite groups. Let $\rho: \Gamma \rightarrow \mathcal{C}$ and let $\psi: \Gamma/\mathcal{K} \rightarrow \mathcal{C}$ be the maps induced by φ . Then*

- (1) $\Gamma_{\infty,\varphi} \cong \Gamma_{\infty,\rho} = (\Gamma/\mathcal{K})_{\infty,\psi}$;
- (2) *the normal closures series $\{\Gamma_{i,\rho}\}$ and the series $\{(\Gamma/\mathcal{K})_{i,\psi}\}$ terminate, hence $\Gamma_{\infty,\varphi} \cong \Gamma_{t_1,\rho} \cong (\Gamma/\mathcal{K})_{t_2,\psi}$, where $\Gamma_{t_1,\rho}$ and $(\Gamma/\mathcal{K})_{t_2,\psi}$ are the terminal members of the respective series;*
- (3) $\Gamma_{\infty,\varphi}$ is finite and $|\Gamma_{\infty,\varphi}| \leq |\Gamma/\mathcal{K}| \cdot g(|\mathcal{C}|)$, where $g(1) = 1$, and for any integer $t \geq 2$,

$$g(t) = t^k, \quad \text{where } k = \frac{1}{2}(\log_p t + 1) \text{ and } p \text{ is the least prime divisor of } t.$$

Proof. Part (1) follows from Proposition 2.6(2) and Corollary 2.10. Then, by definition, $\mathcal{C} = \langle \varphi(\Gamma)^\mathcal{C} \rangle$, so parts (2) and (3) follow from [FS1, Theorem 4.1]. \square

Furthermore we can use Corollary 2.10 to prove the following lemma, which leads to the proof of Theorem 1 of the introduction.

Lemma 2.12. *Assume that $\varphi(\Gamma)$ is subnormal in G and that the series $\{K_i\}$ (see Notation 2.2(2)) terminates. Then $\Gamma_\infty = \Gamma/\mathcal{K}$.*

Proof. Since $\varphi(\Gamma)$ is subnormal in G , we have $\mathcal{C} = \varphi(\Gamma)$. By Proposition 2.6(2), to evaluate Γ_∞ , we may assume that φ is surjective. By [FS1, Corollary 3.9(2)], $\Gamma_2 = \Gamma/[\Gamma, K]$, and $\varphi_2: \Gamma \rightarrow \Gamma_2$ is the canonical map. Iterating on [FS1, Corollary 3.9(2)] we see that $\Gamma_i = \Gamma/K_i$, for all $i \geq 2$, thus $\Gamma_\infty = \Gamma/\mathcal{K}$. \square

Part (2) of the following corollary is Theorem 1 of the introduction.

Corollary 2.13. (1) *If $\varphi(\Gamma)$ is subnormal in G and K is contained in $Z_i(\Gamma)$ for some integer $i \geq 0$ (which holds if φ is injective), then $\Gamma_\infty = \Gamma$;*
 (2) *if Γ and G are nilpotent, then $\Gamma_\infty = \Gamma$.*

Proof. Part (1) is an immediate consequence of Lemma 2.12, since under the hypotheses of (1), $\mathcal{K} = 1$. Then (2) follows from (1). \square

We now turn to the nilpotent quotients of $\Gamma_\infty = \Gamma_{\infty, \varphi}$, and prove Theorem 2 of the introduction.

Proposition 2.14. *Let $k \geq 1$. The map $\varphi_{(\infty, k)}: \Gamma/\gamma_k(\Gamma) \rightarrow \Gamma_\infty/\gamma_k(\Gamma_\infty)$ induced by the canonical map $\varphi_\infty: \Gamma \rightarrow \Gamma_\infty$ is injective. If Γ and G are finite, then it induces an isomorphism $\Gamma/\gamma_\infty(\Gamma) \cong \Gamma_\infty/\gamma_\infty(\Gamma_\infty)$.*

Proof. Let $\psi: \Gamma/\gamma_k(\Gamma) \rightarrow G/\gamma_k(G)$ be the map induced by φ . By naturality (Lemma 2.3) we have the following commutative diagram

$$\begin{array}{ccccc}
 \Gamma & \xrightarrow{\varphi_\infty} & \Gamma_\infty & \xrightarrow{\quad} & G \\
 \downarrow \mu & & \swarrow & \downarrow \eta_\infty & \downarrow \eta \\
 & & \Gamma_\infty/\gamma_k(\Gamma_\infty) & & \\
 \Gamma/\gamma_k(\Gamma) & \xrightarrow[\psi_\infty]{\varphi_{(\infty, k)}} & & (\Gamma/\gamma_k(\Gamma))_{\infty, \psi} & \longrightarrow & G/\gamma_k(G)
 \end{array}$$

where the map η_∞ is obtained from Lemma 2.3, and where ψ_∞ is an isomorphism by Corollary 2.13(2). The diagram is commutative since μ is surjective. Since ψ_∞ is an isomorphism, $\varphi_{(\infty, k)}$ is injective.

Assume that Γ and G are finite. By Proposition 2.6(2) we may assume that $G = \mathcal{C}$, so $G = \langle \varphi(\Gamma)^G \rangle$. Also, by Theorem 2.11(2), the series $\{\Gamma_i\}$ terminates, so $\Gamma_\infty = \Gamma_t$, for some $t \geq 1$, and now $\varphi_\infty = \varphi_t$. By [FS1, Lemma 4.2(1)], $\Gamma_\infty = \Gamma_t = \langle \varphi_t(\Gamma)^{\Gamma_t} \rangle$. Hence the conjugates of the image of $\varphi_t(\Gamma)$ in $\Gamma_t/\gamma_k(\Gamma_t)$ generates it. By [FS1, Lemma 4.4], the image of $\varphi_t(\Gamma)$ in $\Gamma_t/\gamma_k(\Gamma_t)$ equals $\Gamma_t/\gamma_k(\Gamma_t)$. This shows that $\varphi_{(\infty, k)}$ is surjective. \square

3. UNIVERSALITY OF Γ_∞

In this section we prove theorem 5 of the introduction.

Proof. To facilitate the discussion, and in view of the theorem we are now proving, we refer to the factorization $\Gamma \rightarrow \Gamma_{\infty, \varphi} \rightarrow G$ as the *subnormal closure of φ* . When the maps are understood, we refer to the group $\Gamma_{\infty, \varphi}$ itself as the subnormal closure of Γ with respect to G . Notice that the subnormal closure is a functor on maps of groups and thus acts on squares as in equation (2.1) of Lemma 2.3, and respects compositions. We use the following

Notation 3.1. Given a commutative diagram of group homomorphisms:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & G \\ \downarrow = & & \downarrow \eta \\ \Gamma & \xrightarrow{\psi} & H \end{array}$$

we denote by $\bar{\eta}_{\infty}$ the induced map on the subnormal closures:

$$\bar{\eta}_{\infty} : \Gamma_{\infty, \varphi} \rightarrow \Gamma_{\infty, \psi}.$$

Throughout this proof Γ_{∞} denotes $\Gamma_{\infty, \varphi}$. We begin the proof by noting that by Theorem 2.11, Γ_{∞} is finite. Further, by Theorem 2.11, the map $\rho : \Gamma \rightarrow \mathcal{C}$, where \mathcal{C} is as in Notation 2.2(1), satisfies $\Gamma_{\infty} = \Gamma_{t, \rho} = \Gamma_{\infty, \rho}$, for some integer $t \geq 1$. By the same theorem we have a finite tower of normal maps, the first r being normal inclusions leading from the subnormal closure \mathcal{C} to G :

$$(3.1) \quad \Gamma_{\infty} = \Gamma_{t, \rho} \xrightarrow{\overline{\rho_{t-1}}} \Gamma_{t-1, \rho} \cdots \rightarrow \Gamma_{2, \rho} \rightarrow \mathcal{C} = C_r \hookrightarrow C_{r-1} \hookrightarrow \cdots \hookrightarrow G.$$

This implies that the canonical map:

$$l : \Gamma_{\infty} \rightarrow G$$

is a subnormal map, by definition. But by lemma 4.2 of [FS1] the normal closure of the image of the map $\varphi_{\infty} : \Gamma \rightarrow \Gamma_{\infty} = \Gamma_{t, \rho}$, is Γ_{∞} . Since the tower (3.1) is finite and terminates at Γ_{∞} , we have

$$\Gamma^{\varphi_{\infty}} \rightarrow \Gamma_{\infty}$$

is the trivial extension and we take this map as the identity. Of course this implies that the normal closures tower of $\varphi_{\infty} : \Gamma \rightarrow \Gamma_{\infty}$ is constant, so

$$\bar{l}_{\infty} : \Gamma_{\infty, \varphi_{\infty}} \longrightarrow \Gamma_{\infty}$$

is the identity map.

Now we show that the map l is initial among all subnormal factorization maps of φ . Let $\Gamma \xrightarrow{\psi} S \xrightarrow{s} G$ be a factorization of our map φ via a subnormal map $s : S \rightarrow G$. We need to show that there is a unique map $\tilde{s} : \Gamma_{\infty} \rightarrow S$ rendering the following diagram commutative.

$$(3.2) \quad \begin{array}{ccccc} \Gamma & \xrightarrow{\varphi_{\infty}} & \Gamma_{\infty} & \xrightarrow{l} & G \\ \downarrow = & & \downarrow \tilde{s} & & \downarrow = \\ \Gamma & \xrightarrow{\psi} & S & \xrightarrow{s} & G \end{array}$$

To see this we consider the map induced on the subnormal closures by the given subnormal map s :

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \Gamma_{\infty, \psi} & \xrightarrow{\lambda} & S \\ \downarrow = & & \cong \downarrow \bar{s}_{\infty} & & \downarrow s \\ \Gamma & \longrightarrow & \Gamma_{\infty} & \xrightarrow{l} & G \end{array}$$

Here \bar{s}_{∞} is the map induced by naturality of the subnormal closure as in Lemma 2.3, and $\Gamma_{\infty, \psi}$ is the subnormal closure of the map $\psi : \Gamma \rightarrow S$. Since \bar{s}_{∞} is an isomorphism by Proposition 2.4, one gets a well defined map

$$\tilde{s} = \lambda \circ (\bar{s}_{\infty})^{-1} : \Gamma_{\infty} \rightarrow S.$$

To see that this latter map is unique consider any map $\sigma : \Gamma_{\infty} \rightarrow S$ as in diagram 3.2, with \tilde{s} replaced with σ . This map σ induces, by naturality, the following commutative diagram of groups, where the two lower squares do not depend on the choice of σ :

$$\begin{array}{ccccccc} \Gamma & \longrightarrow & \Gamma_{\infty, \varphi_{\infty}} & \xrightarrow{=} & \Gamma_{\infty} & \xrightarrow{l} & G \\ \downarrow = & & \downarrow \bar{\sigma}_{\infty} & & \downarrow \sigma & & \downarrow = \\ \Gamma & \longrightarrow & \Gamma_{\infty, \psi} & \xrightarrow{\lambda} & S & \xrightarrow{s} & G \\ \downarrow = & & \downarrow \tilde{s}_{\infty} & & \downarrow s & & \\ \Gamma & \longrightarrow & \Gamma_{\infty} & \xrightarrow{l} & G & & \end{array}$$

Now we rewrite the map σ in terms of the maps in the given decomposition $\Gamma \rightarrow S \rightarrow G$ alone: We read in the middle upper square: $\sigma = \bar{\sigma}_{\infty} \circ \lambda$. But since:

$$\bar{\sigma}_{\infty} \circ \bar{s}_{\infty} = \overline{(\sigma \circ s)}_{\infty} = \bar{l}_{\infty} = id$$

It follows that both $\bar{\sigma}_{\infty}$, being the inverse to \bar{s}_{∞} , and thus σ itself, are determined by λ —constructed out of ψ and s as claimed. \square

4. EXAMPLES

In this section we give two examples. In both examples we take G to be perfect with $G = \langle \varphi(\Gamma)^G \rangle$. In the first Example 4.1 we assume that Γ and G are finite and that $\ker \varphi \leq Z_{\infty}(\Gamma)$, and we show that $\ker \varphi_{\infty}$ can have arbitrarily large nilpotency class. In the second Example 4.4 we show that if Γ is perfect, then $\Gamma_{\infty} = \Gamma_2$ is the universal φ -central extension of G (see [FS2]).

Example 4.1. In this example we assume that Γ and G are finite and that $\ker \varphi \leq Z_{\infty}(\Gamma)$. The purpose of this example is to show that the nilpotency class of $\ker \varphi_{\infty}$ can be arbitrarily large. We first need the following easy lemma and its corollary.

Lemma 4.2. *Let G be perfect and set $H := \Gamma_\infty$. Then H is a central product $H = H^{(\infty)} \circ Z_\infty(H)$. Here, $H^{(\infty)}$ is the last term of the derived series of H . If the nilpotent residual $\Gamma/\gamma_\infty(\Gamma)$ of Γ has nilpotency class c , then the nilpotency class of $Z_\infty(H)$ is $\leq c + 1$.*

Proof. First, by [FS1, Theorem 4.1], $H = \Gamma_\infty$ is finite, because $G = \langle \varphi(\Gamma)^G \rangle$. Set $L = H^{(\infty)}$ and $\mathcal{Z} = Z_\infty(H)$. There is surjection $H \twoheadrightarrow G$ whose kernel is contained in \mathcal{Z} , and since G is perfect, we get that $H = L\mathcal{Z}$. Since L is perfect, $Z_\infty(L) = Z(L)$, and of course $[L, \mathcal{Z}] \leq Z_\infty(L) = Z(L)$. By the three subgroup lemma we get $[L, \mathcal{Z}] = 1$, and so $H = L \circ \mathcal{Z}$.

Next, by Proposition 2.14, the nilpotent residuals of Γ and H are isomorphic. Clearly $\gamma_\infty(H) = L$ and $\Gamma/\gamma_\infty(\Gamma) \cong H/\gamma_\infty(H) = H/L \cong \mathcal{Z}/(\mathcal{Z} \cap L)$ is nilpotent of class c . Hence, since $\mathcal{Z} \cap L \leq Z(\mathcal{Z})$, we see that the nilpotency class of \mathcal{Z} is $\leq c + 1$. \square

Corollary 4.3. *Let G be perfect, and let c (resp. d) be the nilpotency class of the nilpotent residual of Γ (resp. of $K := \ker \varphi$). Then $\ker \varphi_\infty$ has nilpotency class $\geq d - c - 1$.*

Proof. Let $f: \Gamma_\infty \twoheadrightarrow G$. Then $\ker f \leq Z_\infty(\Gamma_\infty)$. Since $\varphi_\infty \circ f = \varphi$, we see that $\varphi_\infty(K) \leq \ker f \leq Z_\infty(\Gamma_\infty)$. Now $\ker \varphi_\infty \leq K$. By Lemma 4.2, the nilpotency class of $\varphi_\infty(K) \leq c + 1$. Since $K/\ker \varphi_\infty \cong \varphi_\infty(K)$, and the nilpotency class of K is d , the corollary follows. \square

We now construct examples where G is perfect (in fact simple), $\gamma_\infty(\Gamma) = [\Gamma, \Gamma]$, so one has:

- The nilpotent residual of Γ has nilpotency class $c = 1$.
- The nilpotency class d of $\ker \varphi$ is arbitrarily large.

By Corollary 4.3, the nilpotency class of $\ker \varphi_\infty$ is $\geq d - 2$, so it is arbitrarily large.

Let p be a prime, let $n \geq 2$ such that p divides n and let q be a prime power such that p divides $q - 1$. Let H be an image of $\mathrm{SL}_n(q)$ such that $Z(H) \cong \mathbb{Z}_p$ is cyclic of order p . Let B be an elementary abelian group of order p^n , and let Γ be the wreath product $G = H \wr B$. By [M, section 3, p. 282–283], $Z_\infty(\Gamma)$ is of nilpotency class $(n - 1)p + 1$ ([M, Lemma 3.2, p. 283]). Also, $\gamma_\infty(\Gamma) = [\Gamma, \Gamma]$ is isomorphic to a direct product of p^n copies of H ([M, Lemma 3.1, p. 283]). Clearly $\Gamma/Z_\infty(\Gamma)$ is isomorphic to $\mathrm{PSL}_n(q) \wr B$. But $\mathrm{PSL}_n(q) \wr B$ is contained in $G = \mathrm{PSL}_{np^n}(q)$. It follows that there exists a homomorphism $\varphi: \Gamma \rightarrow G$ whose kernel is $Z_\infty(\Gamma)$. Hence G, Γ and $\ker \varphi$ have the claimed properties.

Example 4.4. Suppose Γ is perfect. Then under the above assumptions, G is perfect and $\Gamma_\infty = \Gamma_2$ is a perfect group which is the universal φ -central extension of G .

Indeed, since $G = \langle \varphi(\Gamma)^G \rangle$, and $\varphi(\Gamma)$ is perfect (because Γ is), G is perfect. Also, by [FS1, Lemma 3.3], Γ_2 is a central extension of G , and Γ_2 is generated by $\{\varphi(\Gamma)^g \mid g \in G\}$, so Γ_2 is perfect. The same argument shows that Γ_3 is a perfect central extension of Γ_2 . By [CDFS, Prop. 1.8, p. 637], Γ_3 is a central extension of G . It is easy to check now that $\Gamma_2 = \Gamma_3$, by the universal property of Γ_2 , and that the assertion above holds.

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